# On boundary behavior of generalized quasi-isometries

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**Abstract**

It is established a series of criteria for continuous and homeomorphic extension to the boundary of the so-called lower $Q$-homeomorphisms $f$ between domains in $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, $n \geqslant 2$, under integral constraints of the type $\int \Phi(Q^{n-1}(x))\, dm(x) < \infty$ with a convex non-decreasing function $\Phi : [0, \infty] \to [0, \infty]$. It is shown that integral conditions on the function $\Phi$ found by us are not only sufficient but also necessary for a continuous extension of $f$ to the boundary. It is given also applications of the obtained results to the mappings with finite area distortion and, in particular, to finitely bi-Lipschitz mappings that are a far reaching generalization of isometries as well as quasi-isometries in $\mathbb{R}^n$. In particular, it is obtained a generalization and strengthening of the well-known theorem by Gehring–Martio on a homeomorphic extension to boundaries of quasiconformal mappings between QED (quasiextremal distance) domains.

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**Key words:** mappings with finite area distortion, moduli of families of surfaces, finitely bi-Lipschitz mappings, weakly flat and strongly accessible boundaries.

## 1 Introduction

In the theory of mappings quasiconformal in the mean, integral conditions of the type

$$\int\limits_D \Phi(K(x))\, dm(x) < \infty \tag{1.1}$$

are applied to various characteristics $K$ of these mappings, see e.g. [1], [3], [14], [21]–[26], [38], [39], [41], [46], [48], [53] and [54]. Here $dm(x)$ corresponds to the Lebesgue measure in a domain $D$ in $\mathbb{R}^n$, $n \geqslant 2$. Investigations of classes with the integral conditions (1.1) are also actual in the connection with the recent development of the theory of degenerate Beltrami equations, see e.g. [2], [4], [5], [7]–[11], [15]–[17], [22], [27], [28]–[30], [36], [42]–[45], [47], [52] and the so-called

mappings with finite distortion, see related references e.g. in the monographs [16] and [30].

The present paper is a natural continuation of our previous works [19] and [20], see also Chapters 9 and 10 in the monograph [30], that have been devoted to integral conditions of other types turned out to be useful under the study of mappings with the constraints of the type (1.1).

Recall some definitions. Given a family $\Gamma$ of $k$-dimensional surfaces $S$ in $\mathbb{R}^n$, $n \geqslant 2$, $k = 1, \ldots, n-1$, a Borel function $\varrho : \mathbb{R}^n \to [0, \infty]$ is called **admissible** for $\Gamma$, abbr. $\varrho \in \operatorname{adm}\Gamma$, if

$$\int\limits_S \varrho^k \, d\mathcal{A} \geqslant 1 \tag{1.2}$$

for every $S \in \Gamma$. The **modulus** of $\Gamma$ is the quantity

$$M(\Gamma) = \inf_{\varrho \in \operatorname{adm}\Gamma} \int\limits_{\mathbb{R}^n} \varrho^n(x)\, dm(x) \ . \tag{1.3}$$

We say that a property $P$ holds for **a.e.** (almost every) $k$-dimensional surface $S$ in a family $\Gamma$ if a subfamily of all surfaces of $\Gamma$ for which $P$ fails has the modulus zero.

The following concept was motivated by Gehring's ring definition of quasiconformality in [12]. Given domains $D$ and $D'$ in $\overline{\mathbb{R}^n} = \mathbb{R}^n \cup \{\infty\}$, $n \geqslant 2$, $x_0 \in \overline{D} \setminus \{\infty\}$, and a measurable function $Q : D \to (0, \infty)$, we say that a homeomorphism $f : D \to D'$ is a **lower Q-homeomorphism at the point** $x_0$ if

$$M(f\Sigma_\varepsilon) \geqslant \inf_{\varrho \in \operatorname{adm}\Sigma_\varepsilon} \int\limits_{D \cap R_\varepsilon} \frac{\varrho^n(x)}{Q(x)}\, dm(x) \tag{1.4}$$

for every ring

$$R_\varepsilon = \{x \in \mathbb{R}^n : \varepsilon < |x - x_0| < \varepsilon_0\}, \quad \varepsilon \in (0, \varepsilon_0),\ \varepsilon_0 \in (0, d_0) \ ,$$

where

$$d_0 = \sup_{x \in D} |x - x_0| \ ,$$

and $\Sigma_\varepsilon$ denotes the family of all intersections of the spheres

$$S(r) = S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}, \quad r \in (\varepsilon, \varepsilon_0) \ ,$$

with $D$. The notion can be extended to the case $x_0 = \infty \in \overline{D}$ in the standard way by applying the inversion $T$ with respect to the unit sphere in $\overline{\mathbb{R}^n}$, $T(x) = x/|x|^2$, $T(\infty) = 0$, $T(0) = \infty$. Namely, a homeomorphism $f : D \to D'$ is a **lower $Q$-homeomorphism at** $\infty \in \overline{D}$ if $F = f \circ T$ is a lower $Q_*$-homeomorphism with $Q_* = Q \circ T$ at 0. We also say that a homeomorphism $f : D \to \overline{\mathbb{R}^n}$ is a **lower**

**$Q$-homeomorphism in** $D$ if $f$ is a lower $Q$-homeomorphism at every point $x_0 \in \overline{D}$.

Further we also give applications of results on lower $Q$-homeomorphisms to the mappings with finite area distortion (FAD) and to finitely bi-Lipschitz mappings.

Given domains $D$ and $D'$ in $\mathbb{R}^n$, $n \geqslant 2$, following for [31] we say that a homeomorphism $f : D \to D'$ is of **finite metric distortion**, $f \in \mathrm{FMD}$, if $f$ has $(N)$-property and

$$0 < l(x,f) \leqslant L(x,f) < \infty \quad \text{a.e.} \tag{1.5}$$

where

$$L(x,f) = \limsup_{y \to x} \frac{|f(x) - f(y)|}{|x - y|} ,$$

and

$$l(x,f) = \liminf_{y \to x} \frac{|f(x) - f(y)|}{|x - y|} .$$

Note that a homeomorphism $f : D \to D'$ is of FMD if and only if $f$ is differentiable with $J(x,f) \neq 0$ a.e. and has $(N)$-property, see Remark 3.11 and Corollary 3.14 in [31].

We say that a homeomorphism $f : D \to D'$ has **$(A_k)$-property** if the two conditions hold:

$(A_k^{(1)})$ : for a.e. $k-$dimensional surface $S$ in $D$ the restriction $f|_S$ has $(N)$-property with respect to area;

$(A_k^{(2)})$ : for a.e. $k-$dimensional surface $S_*$ in $D'$ the restriction $f^{-1}|_{S_*}$ has $(N)$-property with respect to area.

We also say that a homeomorphism $f : D \to D'$ is of **finite area distortion in dimension** $k = 1, \ldots, n-1$, $f \in \mathrm{FAD}_k$, if $f \in \mathrm{FMD}$ and has the $(A_k)-$property. Finally, we say that a homeomorphism $f : D \to D'$ is of **finite area distortion**, $f \in \mathrm{FAD}$, if $f \in \mathrm{FAD}_k$ for every $k = 1, \ldots, n-1$. By Lemma 4.1 in [19] every homeomorphism $f \in \mathrm{FAD}_{n-1}$ is a lower $Q$-homeomorphism with $Q(x)$ which is equal to its outer dilatation. It is known, in particular, that every quasiconformal mapping belongs to $\mathrm{FAD}_{n-1}$, see e.g. Theorem 12.6 in [30].

Recall that the **outer dilatation** of a mapping $f : D \to \mathbb{R}^n$, $n \geqslant 2$, at a point $x \in D$ of differentiability for $f$ is the quantity

$$K_O(x,f) = \frac{||f'(x)||^n}{|J(x,f)|}$$

if $J(x,f) \neq 0$, $K_O(x,f) = 1$ if $f'(x) = 0$, and $K_O(x,f) = \infty$ at the rest points. As usual, here $f'(x)$ denotes the Jacobian matrix of $f$ at the point $x$, $J(x,f) = \det f'(x)$ is its determinant and

$$||f'(x)|| = \sup_{h \in \mathbb{R}^n \setminus \{0\}} \frac{|f'(x)h|}{|h|} .$$

A homeomorphism $f : D \to D'$ is called **finitely bi-Lipschitz** if

$$0 < l(x,f) \leqslant L(x,f) < \infty \qquad \forall\, x \in D\ . \tag{1.6}$$

By Theorem 5.5 in [19] every finitely bi-Lipschitz homeomorphism $f$ is of finite area distortion and hence it is a lower $Q$-homeomorphism with $Q(x) = K_O(x,f)$.

## 2 Weakly flat and strongly accessible boundaries

Recall first of all the following topological notion. A domain $D \subset \mathbb{R}^n$, $n \geqslant 2$, is said to be **locally connected at a point** $x_0 \in \partial D$ if, for every neighborhood $U$ of the point $x_0$, there is a neighborhood $V \subseteq U$ of $x_0$ such that $V \cap D$ is connected. Note that every Jordan domain $D$ in $\mathbb{R}^n$ is locally connected at each point of $\partial D$, see e.g. [51], p. 66.

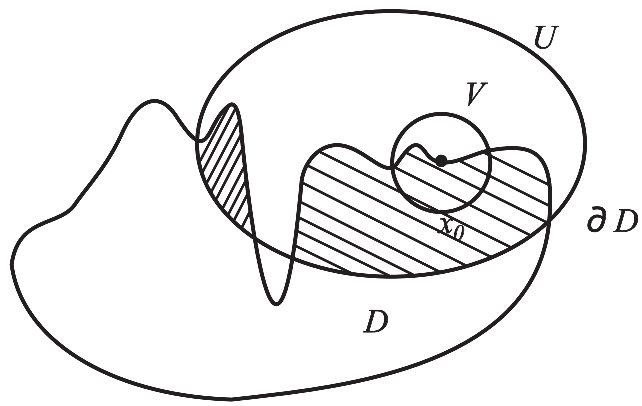


Figure 1.

We say that $\partial D$ is **weakly flat at a point** $x_0 \in \partial D$ if, for every neighborhood $U$ of the point $x_0$ and every number $P > 0$, there is a neighborhood $V \subset U$ of $x_0$ such that

$$M(\Delta(E,F;D)) \geqslant P \tag{2.1}$$

for all continua $E$ and $F$ in $D$ intersecting $\partial U$ and $\partial V$. Here and later on, $\Delta(E,F;D)$ denotes the family of all paths $\gamma : [a,b] \to \overline{\mathbb{R}^n}$ connecting $E$ and $F$ in $D$, i.e. $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in D$ for all $t \in (a,b)$. We say that the boundary $\partial D$ is **weakly flat** if it is weakly flat at every point in $\partial D$.

We also say that a point $x_0 \in \partial D$ is **strongly accessible** if, for every neighborhood $U$ of the point $x_0$, there exist a compactum $E$, a neighborhood $V \subset U$ of $x_0$ and a number $\delta > 0$ such that

$$M(\Delta(E,F;D)) \geqslant \delta \tag{2.2}$$

for all continua $F$ in $D$ intersecting $\partial U$ and $\partial V$. We say that the boundary $\partial D$ is **strongly accessible** if every point $x_0 \in \partial D$ is strongly accessible.

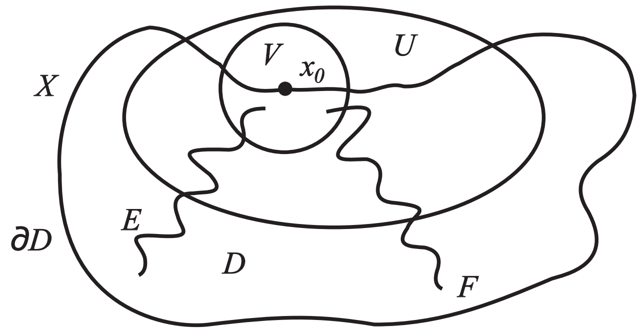


Figure 2.

Here, in the definitions of strongly accessible and weakly flat boundaries, one can take as neighborhoods $U$ and $V$ of a point $x_0$ only balls (closed or open) centered at $x_0$ or only neighborhoods of $x_0$ in another its fundamental system. These conceptions can also in a natural way be extended to the case of $\overline{\mathbb{R}^n}$, $n \geqslant 2$, and $x_0 = \infty$. Then we must use the corresponding neighborhoods of $\infty$.

It is easy to see that if a domain $D$ in $\mathbb{R}^n$, $n \geqslant 2$, is weakly flat at a point $x_0 \in \partial D$, then the point $x_0$ is strongly accessible from $D$. Moreover, it was proved by us that if a domain $D$ in $\mathbb{R}^n$, $n \geqslant 2$, is weakly flat at a point $x_0 \in \partial D$, then $D$ is locally connected at $x_0$, see e.g. Lemma 5.1 in [20] or Lemma 3.15 in [30].

The notions of strong accessibility and weak flatness at boundary points of a domain in $\mathbb{R}^n$ defined in [18] are localizations and generalizations of the corresponding notions introduced in [32]–[33], cf. with the properties $P_1$ and $P_2$ by Väisälä in [49] and also with the quasiconformal accessibility and the quasiconformal flatness by Näkki in [37]. Many theorems on a homeomorphic extension to the boundary of quasiconformal mappings and their generalizations are valid under the condition of weak flatness of boundaries. The condition of strong accessibility plays a similar role for a continuous extension of the mappings to the boundary. In particular, recently we have proved the following significant statements, see either Theorem 10.1 (Lemma 6.1) in [20] or Theorem 9.8 (Lemma 9.4) in [30].

**Proposition 2.1.** *Let $D$ and $D'$ be bounded domains in $\mathbb{R}^n$, $n \geq 2$, $Q : D \to (0, \infty)$ a measurable function and $f : D \to D'$ a lower $Q$-homeomorphism in $D$. Suppose that the domain $D$ is locally connected on $\partial D$ and that the domain $D'$ has a (strongly accessible) weakly flat boundary. If*

$$\int\limits_0^{\delta(x_0)} \frac{dr}{||\,Q||_{\,n-1}(x_0, r)} = \infty \qquad \forall\ x_0 \in \partial D$$

*for some* $\delta(x_0) \in (0, d(x_0))$ *where* $d(x_0) = \sup\limits_{x \in D} |x - x_0|$ *and*

$$|| Q||_{n-1}(x_0, r) = \left( \int\limits_{D \cap S(x_0, r)} Q^{n-1}(x)\, d\mathcal{A} \right)^{\frac{1}{n-1}},$$

*then* $f$ *has a (continuous) homeomorphic extension* $\overline{f}$ *to* $\overline{D}$ *that maps* $\overline{D}$ *(into) onto* $\overline{D'}$.

Here as usual $S(x_0, r)$ denotes the sphere $|x - x_0| = r$.

A domain $D \subset \mathbb{R}^n$, $n \geqslant 2$, is called a **quasiextremal length domain**, abbr. **QED-domain**, see [13], if

$$M(\Delta(E, F; \overline{\mathbb{R}^n}) \leqslant K \cdot M(\Delta(E, F; D)) \tag{2.3}$$

for some $K \geqslant 1$ and all pairs of nonintersecting continua $E$ and $F$ in $D$.

It is well known, see e.g. [49], that

$$M(\Delta(E, F; \mathbb{R}^n)) \geqslant c_n \log \frac{R}{r}$$

for any sets $E$ and $F$ in $\mathbb{R}^n$, $n \geqslant 2$, intersecting all the spheres $S(x_0, \rho)$, $\rho \in (r, R)$. Hence a QED-domain has a weakly flat boundary. One example in [30], Section 3.8, shows that the inverse conclusion is not true even among simply connected plane domains.

A domain $D \subset \mathbb{R}^n$, $n \geqslant 2$, is called a **uniform domain** if each pair of points $x_1$ and $x_2 \in D$ can be joined with a rectifiable curve $\gamma$ in $D$ such that

$$s(\gamma) \leqslant a \cdot |x_1 - x_2| \tag{2.4}$$

and

$$\min_{i=1,2} s(\gamma(x_i, x)) \leqslant b \cdot d(x, \partial D) \tag{2.5}$$

for all $x \in \gamma$, where $\gamma(x_i, x)$ is the portion of $\gamma$ bounded by $x_i$ and $x$, see [34]. It is known that every uniform domain is a QED-domain, but there are QED-domains that are not uniform, see [13]. Bounded convex domains and bounded domains with smooth boundaries are simple examples of uniform domains and, consequently, QED-domains as well as domains with weakly flat boundaries.

A closed set $X \subset \mathbb{R}^n$, $n \geqslant 2$, is called a **null-set of extremal length**, abbr. by **NED-set**, if

$$M(\Delta(E, F; \mathbb{R}^n)) = M(\Delta(E, F; \mathbb{R}^n \backslash X)) \tag{2.6}$$

for any two nonintersecting continua $E$ and $F \subset \mathbb{R}^n \setminus X$.

**Remark 2.1.** It is known that if $X \subset \mathbb{R}^n$ is a NED-set, then

$$|X| = 0 \tag{2.7}$$

and $X$ does not locally separate $\mathbb{R}^n$, i.e.,

$$\dim X \leqslant n-2\ . \tag{2.8}$$

Conversely, if a set $X \subset \mathbb{R}^n$ is closed and

$$H^{n-1}(X) = 0\ , \tag{2.9}$$

then $X$ is a NED-set, see [50]. Note also that the complement of a NED-set in $\mathbb{R}^n$ is a very particular case of a QED-domain.

Here $H^{n-1}(X)$ denotes the $(n-1)$-dimensional Hausdorff measure of a set $X$ in $\mathbb{R}^n$. Also we denote by $C(X,f)$ the **cluster set** of the mapping $f : D \to \overline{\mathbb{R}^n}$ for a set $X \subset \overline{D}$,

$$C(X,f) :\ = \left\{ y \in \overline{\mathbb{R}^n} : y = \lim_{k\to\infty} f(x_k),\ x_k \to x_0 \in X,\ x_k \in D \right\}. \tag{2.10}$$

Note, the conclusion $C(\partial D, f) \subseteq \partial D'$ holds for every homeomorphism $f : D \to D'$, see e.g. Proposition 13.5 in [30].

# 3 The main lemma

For every non-decreasing function $\Phi : [0,\infty] \to [0,\infty]$, the **inverse function** $\Phi^{-1} : [0,\infty] \to [0,\infty]$ can be well defined by setting

$$\Phi^{-1}(\tau) = \inf_{\Phi(t)\geqslant\tau} t\ . \tag{3.1}$$

As usual, here inf is equal to $\infty$ if the set of $t \in [0,\infty]$ such that $\Phi(t) \geqslant \tau$ is empty. Note that the function $\Phi^{-1}$ is non-decreasing, too.

**Remark 3.1.** Immediately by the definition it is evident that

$$\Phi^{-1}(\Phi(t)) \leqslant t \quad \forall\, t \in [0,\infty] \tag{3.2}$$

with the equality in (3.2) except intervals of constancy of the function $\Phi(t)$.

Recall that a function $\Phi : [0,\infty] \to [0,\infty]$ is called **convex** if

$$\Phi(\lambda t_1 + (1-\lambda)t_2) \leqslant \lambda\Phi(t_1) + (1-\lambda)\Phi(t_2)$$

for all $t_1$ and $t_2 \in [0,\infty]$ and $\lambda \in [0,1]$.

In what follows, $\mathbb{B}^n$ denotes the unit ball in the space $\mathbb{R}^n$, $n \geqslant 2$,

$$\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| < 1\}.$$

The following statement is a generalization of Lemma 3.1 from [44].

**Lemma 3.1.** *Let $K : \mathbb{B}^n \to [0, \infty]$ be a measurable function and let $\Phi : [0, \infty] \to [0, \infty]$ be a non-decreasing convex function. Then*

$$\int\limits_0^1 \frac{dr}{r k^{\frac{1}{p}}(r)} \geqslant \frac{1}{n} \int\limits_{eM}^{\infty} \frac{d\tau}{\tau \left[\Phi^{-1}(\tau)\right]^{\frac{1}{p}}} \qquad \forall\, p \in (0, \infty) \tag{3.3}$$

*where $k(r)$ is the average of the function $K(x)$ over the sphere $|x| = r$,*

$$M\colon = \fint_{\mathbb{B}^n} \Phi(K(x))\, dm(x) \tag{3.4}$$

*is the mean value of the function $\Phi \circ K$ over the unit ball $\mathbb{B}^n$.*

**Remark 3.2.** Note that (3.3) under every $p \in (0, \infty)$ is equivalent to

$$\int\limits_0^1 \frac{dr}{r k^{\frac{1}{p}}(r)} \geqslant \frac{1}{n} \int\limits_{eM}^{\infty} \frac{d\tau}{\tau \Phi_p^{-1}(\tau)} \qquad \text{where} \qquad \Phi_p(t)\colon = \Phi(t^p)\ . \tag{3.5}$$

*Proof of Lemma 3.1.* The result is obvious if $M = \infty$ because then the integral in the right hand side in (3.3) is zero. Hence we assume further that $M < \infty$. Moreover, we may also assume that $\Phi(0) > 0$ and hence that $M > 0$ (the case $\Phi(0) = 0$ is reduced to it by approximation of $\Phi(t)$ through cutting off its graph lower the line $\tau = \delta > 0$). Denote

$$t_* = \sup_{\Phi_p(t) = \tau_0} t, \qquad \tau_0 = \Phi(0) > 0. \tag{3.6}$$

Setting

$$H_p(t)\colon = \log\, \Phi_p(t), \tag{3.7}$$

we see that

$$H_p^{-1}(\eta) = \Phi_p^{-1}(e^\eta), \quad \Phi_p^{-1}(\tau) = H_p^{-1}(\log\, \tau). \tag{3.8}$$

Thus, we obtain that

$$k^{\frac{1}{p}}(r) = H_p^{-1}\left(\log \frac{h(r)}{r^n}\right) = H_p^{-1}\left(n \log \frac{1}{r} + \log\, h(r)\right) \qquad \forall\, r \in R_* \tag{3.9}$$

where $h(r)\colon = r^n \Phi(k(r)) = r^n \Phi_p\left(k^{\frac{1}{p}}(r)\right)$ and $R_* = \{r \in (0, 1) : k^{\frac{1}{p}}(r) > t_*\}$. Then also

$$k^{\frac{1}{p}}(e^{-s}) = H_p^{-1}\left(ns + \log\, h(e^{-s})\right) \quad \forall\, s \in S_* \tag{3.10}$$

where $S_* = \{s \in (0,\infty) : k^{\frac{1}{p}}(e^{-s}) > t_*\}$.

Now, by the Jensen inequality and convexity of $\Phi$ we have that

$$\int\limits_0^\infty h(e^{-s})\,ds = \int\limits_0^1 h(r)\,\frac{dr}{r} = \int\limits_0^1 \Phi(k(r))\,r^{n-1}\,dr \leqslant \tag{3.11}$$

$$\leqslant \int\limits_0^1 \left(\fint_{S(r)} \Phi(K(x))\,d\mathcal{A}\right) r^{n-1}dr \leqslant \frac{\Omega_n}{\omega_{n-1}} \cdot M = \frac{M}{n}$$

where we use the mean value of the function $\Phi_p \circ K$ over the sphere $S(r) = \{x \in \mathbb{B}^n : |x| = r\}$ with respect to the area measure. As usual, here $\Omega_n$ and $\omega_{n-1}$ is the volume of the unit ball and the area of the unit sphere in $\mathbb{R}^n$, correspondingly. Then arguing by contradiction it is easy to see that

$$|T| = \int\limits_T ds \leqslant \frac{1}{n} \tag{3.12}$$

where $T = \{s \in (0,\infty) : h(e^{-s}) > M\}$. Next, let us show that

$$k^{\frac{1}{p}}\left(e^{-s}\right) \leqslant H_p^{-1}\left(ns + \log\, M\right) \quad \forall\ s \in (0,\infty) \setminus T_* \tag{3.13}$$

where $T_* = T \cap S_*$. Note that $(0,\infty) \setminus T_* = [(0,\infty) \setminus S_*] \cup [(0,\infty) \setminus T] = [(0,\infty) \setminus S_*] \cup [S_* \setminus T]$. The inequality (3.13) holds for $s \in S_* \setminus T$ by (3.10) because $H_p^{-1}$ is a non-decreasing function. Note also that by (3.6)

$$e^{ns}M > \Phi(0) = \tau_0 \quad \forall\ s \in (0,\infty) \tag{3.14}$$

and then by (3.8)

$$t_* < \Phi_p^{-1}(e^{ns}M) = H_p^{-1}\left(ns + \log\, M\right) \quad \forall\ s \in (0,\infty). \tag{3.15}$$

Consequently, (3.13) holds for $s \in (0,\infty) \setminus S_*$, too. Thus, (3.13) is true.

Since $H_p^{-1}$ is non-decreasing, we have by (3.12) and (3.13) that

$$\int\limits_0^1 \frac{dr}{rk^{\frac{1}{p}}(r)} = \int\limits_0^\infty \frac{ds}{k^{\frac{1}{p}}(e^{-s})} \geqslant \int\limits_{(0,\infty)\setminus T_*} \frac{ds}{H_p^{-1}(ns+\Delta)} \geqslant \tag{3.16}$$

$$\geqslant \int\limits_{|T_*|}^\infty \frac{ds}{H_p^{-1}(ns+\Delta)} \geqslant \int\limits_{\frac{1}{n}}^\infty \frac{ds}{H_p^{-1}(ns+\Delta)} = \frac{1}{n}\int\limits_{1+\Delta}^\infty \frac{d\eta}{H_p^{-1}(\eta)}$$

where $\Delta = \log\, M$. Note that $1 + \Delta = \log\, eM$. Thus,

$$\int\limits_0^1 \frac{dr}{rk^{\frac{1}{p}}(r)} \geqslant \frac{1}{n}\int\limits_{\log\, eM}^\infty \frac{d\eta}{H_p^{-1}(\eta)} \tag{3.17}$$

and, after the replacement $\eta = \log \tau$, we obtain (3.5), see (3.8), and hence (3.3).

Since the mapping $t \mapsto t^p$ for every positive $p$ is a sense-preserving homeomorphism $[0,\infty]$ onto $[0,\infty]$ we may rewrite Theorem 2.1 from [44] in the following form which is more convenient for further applications. Here, in (3.19) and (3.20), we complete the definition of integrals by $\infty$ if $\Phi_p(t) = \infty$, correspondingly, $H_p(t) = \infty$, for all $t \geqslant T \in [0,\infty)$. The integral in (3.20) is understood as the Lebesgue–Stieltjes integral and the integrals in (3.19) and (3.21)–(3.24) as the ordinary Lebesgue integrals.

**Proposition 3.1.** *Let $\Phi : [0,\infty] \to [0,\infty]$ be a non-decreasing function. Set*

$$H_p(t) = \log \Phi_p(t), \quad \Phi_p(t) = \Phi(t^p), \quad p \in (0,\infty). \tag{3.18}$$

*Then the equality*

$$\int_{\delta}^{\infty} H_p'(t)\, \frac{dt}{t} = \infty \tag{3.19}$$

*implies the equality*

$$\int_{\delta}^{\infty} \frac{dH_p(t)}{t} = \infty \tag{3.20}$$

*and (3.20) is equivalent to*

$$\int_{\delta}^{\infty} H_p(t)\, \frac{dt}{t^2} = \infty \tag{3.21}$$

*for some $\delta > 0$, and (3.21) is equivalent to every of the equalities:*

$$\int_{0}^{\delta} H_p\left(\frac{1}{t}\right)\, dt = \infty \tag{3.22}$$

*for some $\delta > 0$,*

$$\int_{\delta_*}^{\infty} \frac{d\eta}{H_p^{-1}(\eta)} = \infty \tag{3.23}$$

*for some $\delta_* > H(+0)$,*

$$\int_{\delta_*}^{\infty} \frac{d\tau}{\tau \Phi_p^{-1}(\tau)} = \infty \tag{3.24}$$

*for some $\delta_* > \Phi(+0)$.*

*Moreover, (3.19) is equivalent to (3.20) and hence (3.19)–(3.24) are equivalent each to other if $\Phi$ is in addition absolutely continuous. In particular, all the conditions (3.19)–(3.24) are equivalent if $\Phi$ is convex and non-decreasing.*

It is easy to see that conditions (3.19)–(3.24) are more weak under more great $p$, see e.g. (3.21). It is necessary to give one more explanation. From the right hand sides in the conditions (3.19)–(3.24) we have in mind $+\infty$. If $\Phi_p(t) = 0$ for $t \in [0, t_*]$, then $H_p(t) = -\infty$ for $t \in [0, t_*]$ and we complete the definition $H_p'(t) = 0$ for $t \in [0, t_*]$. Note, the conditions (3.20) and (3.21) exclude that $t_*$ belongs to the interval of integrability because in the contrary case the left hand sides in (3.20) and (3.21) are either equal to $-\infty$ or indeterminate. Hence we may assume in (3.19)–(3.22) that $\delta > t_0$, correspondingly, $\delta < 1/t_0$ where $t_0 \colon = \sup\limits_{\Phi_p(t)=0} t$, $t_0 = 0$ if $\Phi_p(0) > 0$.

# 4 The main results

Combining Proposition 2.1 and Lemma 3.1 we come to the following statement.

**Theorem 4.1.** *Let $D$ and $D'$ be bounded domains in $\mathbb{R}^n$, $n \geqslant 2$, $D$ be locally connected on $\partial D$ and $D'$ have (strongly accessible) weakly flat boundary. Suppose that $f : D \to D'$ is a lower $Q$-homeomorphism in $D$ with*

$$\int\limits_D \Phi(Q^{n-1}(x))\, dm(x) < \infty \tag{4.1}$$

*for a convex non-decreasing function $\Phi : [0, \infty] \to [0, \infty]$. If*

$$\int\limits_{\delta_0}^{\infty} \frac{d\tau}{\tau \left[\Phi^{-1}(\tau)\right]^{\frac{1}{n-1}}} = \infty \tag{4.2}$$

*for some $\delta_0 > \tau_0 \colon = \Phi(0)$, then $f$ has a (continuous) homeomorphic extension $\overline{f}$ to $\overline{D}$ that maps $\overline{D}$ (into) onto $\overline{D'}$.*

**Corollary 4.1.** *If $D$ and $D'$ are either bounded convex domains or bounded domains with smooth bondaries in $\mathbb{R}^n$, $n \geqslant 2$, then every lower $Q$-homeomorphism $f : D \to D'$ with the conditions (4.1) and (4.2) admits a homeomorphic extension $\overline{f} : \overline{D} \to \overline{D'}$.*

Arguing locally we obtain also the following consequence of Theorem 4.1, see Remark 2.1.

**Corollary 4.2.** *Let $D$ be a domain in $\mathbb{R}^n$, $n \geqslant 2$, and let $X$ be a closed subset of $D$. Suppose that $f$ is a lower $Q$-homeomorphism of $D \setminus X$ into $\overline{\mathbb{R}^n}$ such that*

$$H^{n-1}(X) = 0 = H^{n-1}(C(X,f)) \ . \tag{4.3}$$

*If the the conditions (4.1) and (4.2) hold, then $f$ admits a homeomorphic extension to $D$.*

**Remark 4.1.** Note that the condition (4.2) can be rewritten in the form

$$\int\limits_{\delta_0}^{\infty} \frac{d\tau}{\tau \Phi_{n-1}^{-1}(\tau)} = \infty \qquad \text{where} \quad \Phi_{n-1}(t) \colon = \Phi(t^{n-1}) \ . \tag{4.4}$$

Note also that by Proposition 3.1 the condition (4.4) can be replaced by every of the condition (3.19)–(3.23) under $p = n-1$ and, in particular, the condition (3.21) can be rewritten in the form

$$\int\limits_{\delta}^{\infty} \log\, \Phi(t)\, \frac{dt}{t^{n'}} = +\infty \tag{4.5}$$

for some $\delta > 0$ where $\frac{1}{n'} + \frac{1}{n} = 1$, i.e. $n' = 2$ for $n = 2$, $n'$ is strictly decreasing in $n$ and $n' = n/(n-1) \to 1$ as $n \to \infty$.

*Proof of Theorem 4.1.* Indeed, let us extend the function $Q$ by zero outside of $D$ and set, for fixed $x_0 \in \partial D$,

$$K(x) = Q^{n-1}(x_0 + x d_0) \ , \qquad x \in \mathbb{B}^n$$

with some positive $d_0 < \sup\limits_{z \in D} |z - x_0|$. Then by Lemma 3.1 with the given $K(x)$ and $p = n-1$ we have that

$$\int\limits_0^1 \frac{dr}{r k^{\frac{1}{n-1}}(r)} \geqslant \frac{1}{n} \int\limits_{eM}^{\infty} \frac{d\tau}{\tau \left[\Phi^{-1}(\tau)\right]^{\frac{1}{n-1}}} \tag{4.6}$$

where $k(r)$ is the average of $K(x)$ over the sphere $|x| = r$ and

$$M = \fint\limits_{\mathbb{B}^n} \Phi(K(x))\, dm(x). \tag{4.7}$$

Now, after the replacement $y_0 = x_0 + x d_0$ in (4.7), we have by the condition (4.1) that

$$M \leqslant N \colon = \Phi(0) + \frac{1}{\Omega_n d_0^n} \int\limits_D \Phi(Q^{n-1}(y))\, dm(y) < \infty$$

where $\Omega_n$ is the volume of the unit ball in $\mathbb{R}^n$ and after the replacement $\rho = rd_0$ in the left hand side integral in (4.6) we obtain that

$$\int\limits_0^{d_0} \frac{d\rho}{||Q||_{n-1}(x_0,\rho)} \geqslant \frac{1}{n\omega_{n-1}^{\frac{1}{n-1}}} \int\limits_{eN}^{\infty} \frac{d\tau}{\tau\left[\Phi^{-1}(\tau)\right]^{\frac{1}{n-1}}}$$

where $\omega_{n-1}$ is the area of unit sphere in $\mathbb{R}^n$ and

$$||Q||_{n-1}(x_0,\rho) = \left(\int\limits_{|z-x_0|=\rho} Q^{n-1}(z)\, d\mathcal{A}\right)^{\frac{1}{n-1}}.$$

Note that $N > \Phi(0)$. Thus, we conclude from the condition (4.2) that

$$\int\limits_0^{\delta_0} \frac{d\rho}{||Q||_{n-1}(x_0,\rho)} = \infty. \tag{4.8}$$

This is obvious if $\delta := eN \leqslant \delta_0$. If $\delta > \delta_0$, then

$$\int\limits_{\delta_0}^{\infty} \frac{\delta\tau}{\tau\left[\Phi^{-1}(\tau)\right]^{\frac{1}{n-1}}} = \int\limits_{\delta}^{\infty} \frac{\delta\tau}{\tau\left[\Phi^{-1}(\tau)\right]^{\frac{1}{n-1}}} + \int\limits_{\delta_0}^{\delta} \frac{d\tau}{\tau\left[\Phi^{-1}(\tau)\right]^{\frac{1}{n-1}}}$$

where

$$0 < \int\limits_{\delta_0}^{\delta} \frac{\delta\tau}{\tau\left[\Phi^{-1}(\tau)\right]^{\frac{1}{n-1}}} \leqslant \frac{\log\frac{\delta}{\delta_0}}{\left[\Phi^{-1}(\delta_0)\right]^{\frac{1}{n-1}}} < \infty$$

because $\Phi^{-1}(\delta_0) > 0$.

Finally, by Proposition 2.1 and (4.8) we obtain the statements of Theorem 4.1.

Since quasiconformal mappings are in $\mathrm{FAD}_{n-1}$ (of finite area distortion in dimension $n-1$), see e.g. Theorem 12.6 in [30], and QED-domains have weakly flat boundaries, the following consequence of Theorem 4.1 is a far-reaching generalization of the Gehring-Martio theorem on a homeomorphic extension to boundaries of quasiconformal mappings between QED-domains, cf. [13] and [35].

**Theorem 4.2.** *Let $D$ and $D'$ be bounded domains in $\mathbb{R}^n$, $n \geqslant 2$, with weakly flat boundaries and let $f : D \to D'$ be a homeomorphism in $FAD_{n-1}$. If*

$$\int\limits_D \Phi(K_O^{n-1}(x,f))\, dm(x) < \infty \tag{4.9}$$

*where $\Phi$ is convex non-decreasing function satisfying at least one of the conditions (3.19)–(3.24) under $p = n-1$, in particular, (4.2) or (4.5), then $f$ can be extended to a homeomorphism $\overline{f}$ of $\overline{D}$ onto $\overline{D'}$.*

In turn, since finitely bi-Lipschitz homeomorphisms are of finite area distortion in dimension $n-1$, see e.g. Theorem 5.5 in [19], we have also the following consequence.

**Corollary 4.3.** *Every finitely be-Lipschitz homeomorphism $f : D \to D'$ under the hypothesis of Theorem 4.2 admits a homeomorphic extension to the closure of the domains $D$ and $D'$.*

**Remark 4.2.** If the domain $D$ is not bounded, then it must be used the spherical volume $dV(x) = dm(x)/(1+|x|^2)^n$ instead of the Lebesgue measure $dm(x)$ in the above conditions (4.1) and (4.9).

# 5 Necessary conditions for extension

**Theorem 5.1.** *Let $\varphi : [0,\infty] \to [0,\infty]$ be a convex non-decreasing function such that*

$$\int\limits_{\delta_*}^{\infty} \frac{d\tau}{\tau\varphi^{-1}(\tau)} < \infty \tag{5.1}$$

*for some $\delta_* \in (\tau_0, \infty)$ where $\tau_0 := \varphi(0)$. Then for every $n \geqslant 2$ there is a diffeomorphism $f$ of the punctured unit ball $\mathbb{B}^n \setminus \{0\}$ onto a ring $\mathfrak{R} = \{x \in \mathbb{R}^n : 1 < |x| < R\}$ such that*

$$\int\limits_{\mathbb{B}^n} \varphi(K_O(x,f))\, dm(x) < \infty \tag{5.2}$$

*but $f$ cannot be extended by continuity to $0$.*

By the known criterion of convexity, see e.g. Proposition 5 in I.4.3 of [6], the inclination $[\varphi(t) - \varphi(0)]/t$ is non-decreasing. By (5.1) the function $\varphi$ cannot be constant. Thus, the proof of Theorem (5.1) is reduced to the following statement.

**Lemma 5.1.** *Let $\varphi : [0,\infty] \to [0,\infty]$ be a non-decreasing function such that*

$$\varphi(t) \geqslant C \cdot t \qquad \forall\, t \in [T,\infty] \tag{5.3}$$

*for some $C > 0$ and $T \in (0,\infty)$ and (5.1) holds. Then for every $n \geqslant 2$ there is a diffeomorphism $f$ of the punctured unit ball $\mathbb{B}^n \setminus \{0\}$ onto a ring $\mathfrak{R} = \{x \in \mathbb{R}^n : 1 < |x| < R\}$ such that (5.2) holds but $f$ cannot be extended by continuity to $0$.*

*Proof.* Note that by the condition (5.1)

$$\int\limits_{\delta}^{\infty} \frac{d\tau}{\tau\varphi^{-1}(\tau)} < \infty \qquad \forall\, \delta \in (\tau_0, \infty) \tag{5.4}$$

because $\varphi^{-1}(\tau) > 0$ for all $\tau > \tau_0$ and $\varphi^{-1}(\tau)$ is non–decreasing. Then applying the linear transformation $\alpha\varphi + \beta$ with $\alpha = 1/C$ and $\beta = T$, see e.g. (3.21), we may assume that

$$\varphi(t) \geqslant t \qquad \forall\, t \in [0, \infty)\ . \tag{5.5}$$

Of course, we may also assume that $\varphi(t) = t$ for all $t \in [0, 1)$ because the values of $\varphi$ in $[0, 1)$ give no information on $K_O(x, f) \geqslant 1$. It is clear (5.4) implies that $\varphi(t) < \infty$ for all $t < \infty$, see the criterion (3.21), cf. (3.24).

Now, note that the function $\Psi(t) :\ = t\varphi(t)$ is strictly increasing, $\Psi(1) = \varphi(1)$ and $\Psi(t) \to \infty$ as $t \to \infty$. Hence the functional equation

$$\Psi(K(r)) = \left(\frac{\gamma}{r}\right)^2 \qquad \forall\, r \in (0, 1]\ , \tag{5.6}$$

where $\gamma = \varphi^{1/2}(1) \geqslant 1$, is well solvable with $K(1) = 1$ and a strictly decreasing continuous $K(r)$, $K(r) < \infty$, $r \in (0, 1]$, and $K(r) \to \infty$ as $r \to 0$. Taking the logarithm in (5.6), we have that

$$\log\, K(r) + \log\, \varphi(K(r)) = 2\, \log\, \frac{\gamma}{r}$$

and by (5.5) we obtain that

$$\log\, K(r)\ \leqslant\ \log\, \frac{\gamma}{r}\ ,$$

i.e.,

$$K(r)\ \leqslant\ \frac{\gamma}{r}\ . \tag{5.7}$$

Then by (5.6)

$$\varphi(K(r))\ \geqslant\ \frac{\gamma}{r}$$

and by (3.2)

$$K(r)\ \geqslant\ \varphi^{-1}\left(\frac{\gamma}{r}\right)\ . \tag{5.8}$$

Next, we define the following mapping in the unit ball $\mathbb{B}^n$:

$$f(x) = \frac{x}{|x|}\, \varrho(|x|)$$

where

$$\varrho(t)\ =\ \exp\{I(t)\}, \quad I(t)\ =\ \int\limits_0^t \frac{dr}{rK(r)}\ .$$

By (5.8)

$$I(t) \;=\; \int\limits_{0}^{t} \frac{dr}{rK(r)} \;\leqslant\; \int\limits_{0}^{t} \frac{dr}{r\varphi^{-1}\left(\frac{\gamma}{r}\right)} \;=\; \int\limits_{\frac{\gamma}{t}}^{\infty} \frac{d\tau}{\tau\varphi^{-1}(\tau)} \qquad \forall\, t \in (0,1]$$

where $\gamma/t \geqslant \gamma \geqslant 1 > \varphi(0) = 0$. Hence by the condition (5.4)

$$I(t) \;\leqslant\; I(1) \;=\; \int\limits_{0}^{1} \frac{dr}{rK(r)} \;<\; \infty \qquad \forall\, t \in (0,1]\,. \tag{5.9}$$

Note that $f \in C^1\left(\mathbb{B}^n \setminus \{0\}\right)$ because $K(r)$ is continuous and, consequently, $f$ is locally quasiconformal in $\mathbb{B}^n \setminus \{0\}$.

The tangent and radial distortions under the mapping $f$ on the sphere $|\,x| = \rho$, $\rho \in (0,1)$, are easy calculated

$$\delta_\tau(x) \;=\; \frac{|\,f(x)|}{|\,x|} \;=\; \frac{\exp\left\{\int\limits_0^\rho \frac{dr}{rK(r)}\right\}}{\varrho}\,,$$

$$\delta_r(x) \;=\; \frac{\partial|\,f(x)|}{\partial|\,x|} \;=\; \frac{\exp\left\{\int\limits_0^\rho \frac{dr}{rK(r)}\right\}}{\rho K(\rho)}$$

and we see that $\delta_r(x) \leqslant \delta_\tau(x)$ because $K(r) \geqslant 1$. Consequently, by the spherical symmetry we have that

$$K_O(x,f) \;=\; \frac{\delta_\tau^n(x)}{\delta_\tau^{n-1}(x)\cdot\delta_r(x)} \;=\; \frac{\delta_\tau(x)}{\delta_r(x)} \;=\; K(|x|)$$

at all points $x \in \mathbb{B}^n \setminus \{0\}$, see e.g. Subsection I.4.1 in [40]. Thus, by (5.6)

$$\int\limits_{\mathbb{B}^n} \varphi(K_O(x,f))\,dm(x) \;=\; \int\limits_{\mathbb{B}^n} \varphi(K(|x|))\,dm(x) \;=$$

$$=\; \omega_{n-1}\int\limits_{0}^{1} \frac{\Psi(K(r))}{rK(r)}\, r^n dr \;\leqslant\; \gamma^2\omega_{n-1}\int\limits_{0}^{1} \frac{dr}{rK(r)} \;\leqslant\; M\!: \;=\; \gamma^2\omega_{n-1} I(1) \;<\; \infty\,.$$

On the other hand, along every radial line $x/|x| = \eta \in \mathbb{R}^n$, $|\eta| = 1$, we have that $f(x) \to \eta$ as $|x| \to 0$, i.e. we have no determinated limit of $f$ under $x \to 0$. It is easy to see that

$$\lim_{x\to 0} |f(x)| = \lim_{t\to 0} \varrho(t) = e^0 = 1\,, \tag{5.10}$$

i.e. $f$ maps the punctured ball $\mathbb{B}^n \setminus \{0\}$ onto the ring $1 < |y| < R = e^{I(1)}$.

**Remark 5.1.** Note that $f$ in the example under the proof of Theorem 5.1 (Lemma 5.1) is finitely bi-Lipschitz and hence of finite area distortion and, consequently, it is a lower $Q$-homeomorphism with $Q(x) = K_O(x, f)$, that the domains $D$ and $D'$ have weakly flat boundaries, the condition (5.2) holds but $f$ cannot be extended by continuity to the boundary. Thus, the condition (4.2) in Theorem 4.1 is necessary because if a function $\Phi$ is convex, then the function $\varphi = \Phi_{n-1}$, where $\Phi_{n-1}(t) = \Phi(t^{n-1})$, is so. Recall that $\Phi_{n-1}^{-1} = [\Phi^{-1}]^{\frac{1}{n-1}}$.

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